Numbers as Dual Entities: A Conceptual Framework for Abstract Values, Physical Instantiations, and Cross-Layer Relativity

Bas Böttcher

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Abstract

Numbers are typically treated as purely abstract objects, independent of their physical realization. In practice, however, every use of a number requires a physical substrate, such as ink, sound, electronic states, or neural patterns. This paper develops a conceptual model in which a number-in-use is represented as a dual entity $([n],[x])$ consisting of an abstract value $[n]$ and a physical instantiation $[x]$. The model clarifies how abstract operations interact with physical constraints, explains why certain mathematical expressions (such as division by zero) lack physical instantiation, and introduces replication as a fundamental operation acting on the physical layer.

A key contribution is the development of cross-layer relativity, the bidirectional relationship between abstraction and physicality. Abstract magnitude influences physical complexity ($n \to x$), while physical capacity and noise constrain which abstract values can be expressed ($x \to n$). The paper also sketches a composite metric for evaluating the stability and robustness of physical instantiations, drawing on work in information theory, thermodynamics, cognition, and semiotics.

In addition to the core model, the paper provides historical context, implications for information theory, philosophical interpretation, formal extensions, and a discussion of open problems. The aim is modest: to offer a structured way of analyzing how numerical meaning exists in real cognitive, social, and technological systems.

1 Introduction

Mathematics is often regarded as existing in a purely abstract realm, insulated from physical contingency. Whether one adopts Platonism, structuralism, or formalism, numbers themselves are usually considered independent of the ink, pixels, sound waves, or neural circuits through which they are expressed.

Nevertheless, any real-world use of numbers depends on physical substrates. A chalk mark on a blackboard, a sequence of transistors in a CPU, the neural activation pattern underlying a thought—all instantiate numbers physically. This observation motivates the idea that the number-in-use is a compound of abstraction and physical instantiation.

We formalize this using the notation

$$([n], [x]) = (\text{abstract value}, \text{physical instantiation}).$$

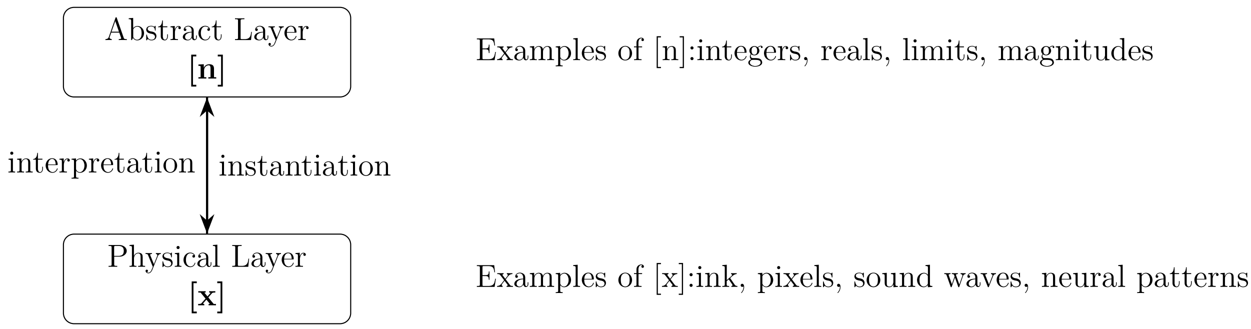

Figure 1: The dual structure of a number-in-use as a pair $([n], [x])$. The abstract value $[n]$ requires a physical instantiation $[x]$, while $[x]$ requires interpretation to recover $[n]$.

This dual structure leads naturally to questions such as: What does it mean for a numerical operation to be physically instantiable? Why is division by zero not only abstractly undefined but physically impossible? How do noise, information loss, and thermodynamic cost constrain numerical meaning?

This framework offers a tool for analyzing how numbers participate in physical systems.

2 Historical Context

2.1 Ancient and Classical Traditions

The Pythagoreans mixed numerology with geometry; Euclid relied on diagrams as physical instantiations of geometric relations. Aristotle distinguished between form and matter—an early echo of abstraction vs. instantiation.

2.2 Early Modern Developments

Galileo tied numbers to measurement. Descartes' analytic geometry linked spatial forms to algebraic abstraction. Leibniz's binary system had practical motivations in physical implementation.

2.3 Nineteenth and Twentieth Century Shifts

Frege and Peano abstracted arithmetic; Hilbert formalized mathematical systems. Meanwhile, Shannon, Turing, and semioticians emphasized the physicality of information and computation.

2.4 Information is Physical

Landauer established an energy cost for bit erasure. Bennett developed reversible computation. Wheeler proposed "it from bit." These arguments link numeric representation to physics.

2.5 Twenty-First Century Trends

Neuroscience, distributed systems, blockchain technologies, and machine learning all demonstrate the intimate link between abstraction and physical instantiation.

3 Numbers as Dual Entities $([n], [x])$

3.1 Formal Definition

A number-in-use is the pair $([n], [x])$, where $[n]$ is the abstract value and $[x]$ its physical realization.

3.2 The Abstract Layer $[n]$

n may be an integer, real, symbolic expression, infinite object, or algebraic structure. Mathematics manipulates $[n]$ independently of instantiation.

3.3 The Physical Layer $[x]$

The physical layer includes material substrate, energy, temporal persistence, waveform form, interpretive mechanisms, and noise.

3.4 Immediate Consequences

Abstraction alone is insufficient for real-world use; two identical abstract numbers may differ physically; some abstract operations lack physical instantiation; replication acts on $[x]$ alone.

3.5 Examples

A printed "7," a RAM-stored integer, a spoken number, a neural representation, and a blockchain hash are all examples of $([n], [x])$ pairs.

4 Axioms Governing the Framework

4.1 Axiom 1: Dual Representation

Every number-in-use is a dual entity $([n], [x])$.

Example: Writing "5" on paper instantiates $([5], [\textit{ink} + \textit{paper} + \textit{illumination} + \textit{decoding}])$.

4.2 Axiom 2: Strict Equality

$$([n_1], [x_1]) = ([n_2], [x_2]) \Leftrightarrow n_1 = n_2 \text{ and } x_1 = x_2.$$

Example: Two USB sticks storing the number 42 differ physically at the atomic scale, so $([42], [x_A]) \neq ([42], [x_B])$.

4.3 Axiom 3: Instantiability of Operations

Operations must be both abstractly defined and physically realizable.

Division by Zero. 12/0 is abstractly undefined and physically impossible because instantiating "0" already requires non-zero material.

Example: "0" is already "something" on the x-layer. That is why a division by zero can never exist.

4.4 Axiom 4: Limits and Approximation

Physical systems can only approximate limit states.

Example: A speaker can approach silence but cannot reach perfect zero amplitude due to thermal noise.

4.5 Axiom 5: Infinity and Instantiation

Symbols for infinity can be instantiated, but infinite quantities cannot.

Example: The glyph "∞" is a finite ink pattern; it is not an infinite object.

4.6 Axiom 6: Replication Operator

Replication duplicates physical instantiation while preserving value:

$$\text{REPLICATE}_k([n], [x]) = \sum_{i=1}^{k} ([n], [x_i]).$$

Example: Speaking "one" produces multiple neural instantiations $([1], [x_i])$ in each listener.

4.7 Axiom 7: Composite Nature of $[x]$

x includes energy, material form, temporal stability, noise susceptibility, and interpretability.

Example: The spoken "seven" depends on pressure waves, vocal tract shape, and listener decoding.

4.8 Axiom 8: Uniqueness of Instantiations

Physical instantiations are never perfectly identical.

Example: Two printed "3"s differ in microscopic ink distribution.

4.9 Axiom 9: Interpretive Requirement

Decoding must map $[x]$ to a specific $[n]$.

Example: OCR misreading a "9" as a "4" shows that $([n], [x])$ requires interpretation.

5 Applications

Digital replication, blockchain anchoring, identity systems, spoken numbers, and measurement devices demonstrate the necessity of the dual framework.

6 Cross-Layer Relativity

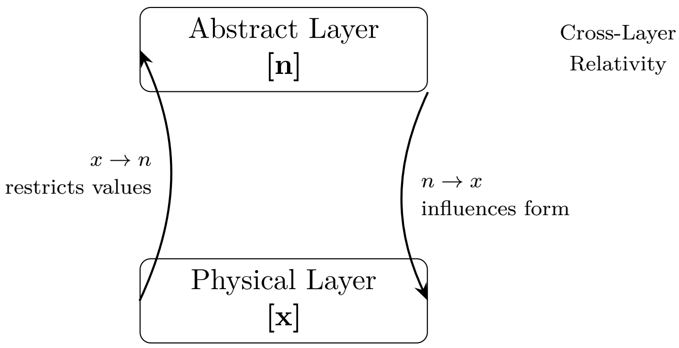

Figure 2: Cross-layer relativity: abstract structure affects physical instantiation ($n \to x$), while physical constraints limit representable abstractions ($x \to n$).

6.1 $n \to x$ Relativity

Abstract properties shape physical instantiation: magnitude, representational complexity, and cryptographic difficulty influence cost.

6.2 $x \to n$ Relativity

Noise, finite precision, and cognitive limits restrict available abstract numbers.

6.3 Formal Principle

For all $([n], [x])$, abstract properties constrain instantiation and physical properties constrain abstraction.

7 Measuring the Physical Layer $[x]$

7.1 Dimensions

Energy stability, temporal persistence, durability, noise resistance, redundancy, interpretability.

7.2 Instantiation Robustness Index

A composite metric conceptualizing robustness.

8 Philosophical Implications

8.1 Abstract vs. Physical

Abstraction is conceptually autonomous but operationally dependent on instantiation.

8.2 Semiotics

$([n], [x])$ parallels signified/signifier but adds physical nuance.

8.3 Cognition

Neural instantiation limits abstraction.

8.4 Information Ontology

"Information is physical" aligns with the dual model.

8.5 Infinite Objects

Infinite abstractions cannot be instantiated fully.

9 Formal Extensions

9.1 Algebra of Instantiations

Some numerical operations fail to instantiate physically, even when they are easily expressible as formal symbols.

Example: A digital device can add two integers, multiply them, or store them in memory, but it cannot instantiate division by zero. The very existence of the device—its hardware, energy supply, and internal states—already presupposes a non-empty physical substrate. An operation that would require "dividing by nothing" cannot be realized in such a system, because the system itself is proof that absolute nothingness is not present.

9.2 Replication as a Primitive

Replication is physically essential yet absent from pure mathematics.

Example: There is no formal symbol "make two copies of 17," yet computers do so constantly.

9.3 Interpretive Maps

Decoding maps physical states to possible abstract values:

$$\text{Decode} : X \to P(N).$$

Example: OCR may interpret a noisy "3" as $\{3, 8, 5\}$.

9.4 Noise Operators

Noise modifies $[x]$, sometimes altering the decoded $[n]$.

Example: A flipped bit may turn $([7], [x])$ into $([15], [x'])$.

9.5 Topological Modeling

Physical instantiations may cluster in high-dimensional spaces.

Example: Different spoken "seven"s cluster in acoustic feature space.

9.6 Probabilistic Instantiation

Some instantiations are distributions:

$$([n], [X_{\text{dist}}]).$$

Example: Human memory of "14" drifts probabilistically.

9.7 Representational Equivalence

Two instantiations may be equivalent if they encode the same n with similar robustness.

Example: Two high-resolution scans of "8" differ microscopically but decode identically.

10 Open Problems

Formalizing $[x]$, linking to computability, thermodynamic limits, robustness calibration, cognitive boundaries, ML constraints, real analysis vs. physical limits, philosophical unification.

11 Conclusion: Numbers Between Abstraction and Reality

This paper proposes that numbers-in-use be understood as dual entities $([n], [x])$. Mathematics focuses on $[n]$, but all real numerical practice requires $[x]$. Cross-layer relativity shows how abstraction and instantiation constrain each other.

Two "touching points" reveal the deepest interactions:

Replication. Mathematics lacks a replication operator, yet mathematical practice depends entirely on replicating physical instantiations. Mathematics is conceptually autonomous but operationally dependent on $[x]$.

Division by Zero. The moment "0" is instantiated physically, it becomes a something, not a nothing; thus 12/0 is not only abstractly undefined but physically impossible and ontologically contradictory.

Numbers are dual citizens of the abstract and physical worlds. The framework clarifies how they persist, degrade, propagate, and acquire meaning. Its goal is modest: to provide language for analyzing interactions between mathematics and the physical, cognitive, and informational systems that make numerical reasoning possible.